Dear Chance News

I'm a musician with a strong interest in mathematics, and I
recently performed a piece which involved probability as a
kind of essential element to the work. The piece raises a
probability question (which I don't know the answer to)
that I thought would be fun to readers of this newsletter.

The piece was by Seattle composer David Mahler, and it
was a trio for mandolin, flute, and piano. It was called
"Short of Success." It was part of a larger work called
"After Richard Hugo", for five musicians. The trio was
based on the idea that one should embrace lack of perfection
as a necessary component of poetry, but nonetheless
strive for perfection.

Here's the way the piece worked. There were nine single
pages of score, each a single melody. Each page was a
slightly different version of every other page.Each of
the three musicians had the same set of nine pages. Before
the performance, each of the three musicians "randomly"
rearranged their pages, independently of the other two
musicians. We then played each page, in unison, until
we heard a "discrepancy." At that point, we stopped
and moved on to the next page. The instructions for
the piece were that if we ever played the same page
(which would have resulted in a single unison melody),
the person who started that particular page (each new
page is cued by one of the musicians) was supposed
to shrug their shoulders, and say, without enthusiasm:
"success".

My question is: what are the odds of that actually
occurring? Needless to say, in four or five performances
of the work, and in maybe 20-30 times rehearsing
it, it never occurred.

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Let's first formalize Larry's question. We assume that the composer labels the 9 versions from 1 to 9. Each player recieves a copy of these nine versions. They mix up their copies and play them in the resulting order. The numbers on the music of a player in the order the versions are played is a random permutation of the numbers from 1 to 9. If the resulting three permuations have the same number in a particular position, this is called a *fixed point* of the three permutations. The players will have success if there is at least fixed point in the three permutations. For example, if the labels in the order they were played are

piano	2	4	6	5	9	3	7	1	8
mandalin	1	6	8	3	2	9	7	4	5
flute	5	8	1	6	9	3	7	4	2

the trio would have success on the 7th run through of the piece.

Since Larry also mentions a larger composition with 5 instruments we will generalize the problem by assuming that there are m players and each player has n versions of the composition. It turns out to be easier to find the probability that there is no fixed-point, i.e., the players fail to play a common version of the piece.

Let E_i be the event that "i" is a fixed point. Let $\bar{E}_i$ be the event that "i" is not a fixed point

Let $f(n, m)$ be the probability of failure. Then the probability of success will be $1 - f(n, m)$. The probability of failure is the probability that their are no fixed points which is:

$$f(n, m) = P(\bar{E}_1 \bar{E}_2 \cdots \bar{E}_n)$$

Using the principle of inclusion-exclusion we have:

$$f(n, m) = 1 - \sum_i P(E_i) + \sum_{i<j} P(E_i E_j) - \sum_{i<j<k} P(E_i E_j E_k) + \cdots (-1)^n P(E_1 E_2 \cdots E_n)$$

We now calculate the sums in this expression. We illustrate the computation in terms of the third sum $\sum_{i<j<k} P(E_i E_j E_k)$. There are $\binom{n}{3} = \frac{n}{3!(n-3)!}$ ways to choose $i < j < k$. For the events $E_i E_j E_k$ to occur, the permutations for the non-piano players must have the same numbers in positions i,j and k. There are n choices for the number at position i, and then $n - 1$ choices for the number at position j and finally $n - 2$ choices for the number

at position k. Thus there are $n(n - 1(n - 2) = \frac{n!}{(n-3)}!$ possibilities for the numbers at i, j, k. For each of these choices there are $(n - 3)!$ possibilities for the numbers at the other positions. The total number of possibilities for the three permutations is $n!^3$.Putting all this together we have

$$\sum_{i<j<k} P(E_i E_j E_k) = \frac{\frac{n}{3!(n-3)!} \frac{n!}{(n-3}!(n-3)!}{n!} = \frac{1}{3!} \left(\frac{(n-3)!}{n!} \right)^{m-2}$$

Carrying out a similar computation for each of the terms and putting these in the above inclusion-exclusion expression we have

$$f(n, m) = \sum_{j=0}^{n} (-1)^j \frac{1}{j!} \left(\frac{(n-j)!}{n!} \right)^{m-2}$$

Note that the case of two players simplifies to:

$$f(n, m) = \sum_{j=0}^{n} (-1)^j \frac{1}{j!}$$

Recall that

$$e^x = \sum_{0}^{\infty} \frac{x^j}{j!}$$

Thus for a composition for 2 players, the probability of not succeeding approaches $1/e = 0.367879$ as $n \to \infty$ and the probability of succeeding approaches $1 - 1/e = 0.632121\ldots$

The case m $= 2$ is equivalent to one of the oldest problems in probability theory now called the "hat-check" problem. In this version of the problem, n men check their hats in a restaurant and the hats get all scrambled up before they are returned. What is the probability that at least one man get his own hat back? What is remarkable about this problem is that the answer is essentially constant $1 - 1/e = 0.632121\ldots$ for any number of men greater than 8.

Here are the probabilities for success with 2 players, 3 players, and 5 players when the number versions varies from 2 to 10.

n	$m = 2$	$m = 3$	$m = 5$
2	0.5	0.25	0.0625
3	0.666667	0.277778	0.0354938
4	0.625	0.213542	0.0153447
5	0.631944	0.1775	0.00793825
6	0.633333	0.151283	0.00461121
7	0.631944	0.131699	0.00290872
8	0.632143	0.116544	0.00195028
9	0.632121	0.104484	0.0013704
10	0.632121	0.0946679	0.000999315

Note that when we have 5 players, the probability of success decreases rapidly. The best probability of success in this case would be a 6 percent chance of success when there are only 2 versions of the piece. There would be only a 1.5 percent chance of success with 4 versions and with 9 versions the players would probability never have success. We see that for Larry's question (m = 3, n = 9) the answer is that the probability of success is .10445. Thus Larry's group should expect to succeed about 1 in 10 times they rehearse or play "Short of Success." Larry expressed surprise with this result since they played or rehearsed this piece about 20 times and performed it five times and had never succeeded. This could be caused by chance (there is about a 10 percent chance of it happening) or by a "false-negative" results.

This is an interesting problem to use to discuss the concepts of "false positive" and "false negative". A false positive result could occur if on a particular run through two players have the same version, the third player has a version which is very close to their version and the difference is simply not noticed. A false negative would occur if on a particular run through they all have the same version but a player hits a wrong note which is interpreted as a difference in his version. The false-positive and false-negative rates could be estimated if the players would keep a record of their permutations and what actually happened when they played the piece.

Larry responded to this by writing:

Re: the false positive and false negative. That, in fact, is integral, I think, to the musical notion of the piece. It's fairly hard for three musicians to always play perfectly in unison without making a mistake (it's a reasonably difficult page of music), and we had LOTS of situations where we weren't sure if it was "us" or the "system." That not only confirms what you hypothesize (that we may have, in fact, "hit it" several times without realizing it), but it is also very much, I think, part of the aesthetic of the work,

which investigates the notions of success, perfection, and failure in wonderful ways. Your formulation of the problem gives some nice added richness (or perhaps I should say, resonance) to the piece itself which I'm sure David is enjoying immensely. I would say (in our defense, since musicians never want to admit to clams) that most of the time when that happened, and we were suspicious, we asked each other "which number did you have up on the stand?" and every single time (strangely), we had different ones. But I can't swear that we ALWAYS confirmed it in this way.

Thanks again, the description is great, the answer fascinating, and completely changes my perspective of a piece that I just performed a number of times!